

ON A SCHUR-LIKE PROPERTY FOR SPACES OF MEASURES

SANDER C. HILLE, TOMASZ SZAREK, DANIEL T.H. WORM, AND MARIA A. ZIEMLAŃSKA

ABSTRACT. A Banach space has the Schur property when every weakly convergent sequence converges in norm. We prove a Schur-like property for measures: if a sequence of finite signed Borel measures on a Polish space is such that it is bounded in total variation norm and such that for each bounded Lipschitz function the sequence of integrals of this function with respect to these measures converges, then the sequence converges in dual bounded Lipschitz norm or Fortet-Mourier norm to a measure. Moreover, we prove three consequences of this result: the first is equivalence of concepts of equicontinuity in the theory of Markov operators, the second is the derivation of weak sequential completeness of the space of signed Borel measures on Polish spaces from our main result and the third concerns conditions for the coincidence of weak and norm topologies on sets of measures that are bounded in total variation norm with additional properties.

1. INTRODUCTION

A Banach space X has the *Schur property* if every weakly convergent sequence in X is norm convergent (e.g. [1], Definition 2.3.4). The sequence space ℓ^1 has the Schur property (cf. [1], Theorem 2.3.6). Here we consider the vector space of finite signed Borel measures $\mathcal{M}(S)$ on a Polish space S . Once a metric d has been fixed that metrizes S as a complete separable metric space, $\mathcal{M}(S)$ can be embedded into the dual space $\text{BL}(S)^*$ of the Banach space $\text{BL}(S)$ of bounded Lipschitz functions on S for d . Denote this space by $\mathcal{M}(S)_{\text{BL}}$. It comes equipped with the dual norm $\|\cdot\|_{\text{BL}}^*$ of $\text{BL}(S)^*$. It is known that generally $\mathcal{M}(S)_{\text{BL}}$ is not complete, unless S is uniformly discrete ([14], Theorem 3.11). Moreover, $\mathcal{M}(S)_{\text{BL}}^* \simeq \text{BL}(S)$, cf. [14], Theorem 3.7.

Our main result is a proof of the following *Schur-like property*:

Theorem (Schur-like property). *Let (S, d) be a complete separable metric space. Let $(\mu_n) \subset \mathcal{M}(S)$ be such that $\sup_n \|\mu_n\|_{\text{TV}} < \infty$. If for every $f \in \text{BL}(S)$ the sequence $\langle \mu_n, f \rangle$ converges, then there exists $\mu \in \mathcal{M}(S)$ such that $\|\mu_n - \mu\|_{\text{BL}}^* \rightarrow 0$ as $n \rightarrow \infty$.*

This is almost the statement that $\mathcal{M}(S)_{\text{BL}}$ has the Schur property, except for the condition of bounded total variation norm (or the completeness of $\mathcal{M}(S)_{\text{BL}}$). For various relationships

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between weak and norm convergence of sequences of measures the reader should consult [7]. There, on page 261, Dudley mentions that he has a proof for the main theorem apparently, halve a century ago, but: ‘... *this result seems irrelevant to the main purpose of this paper* [i.e. [7]], *and the argument seems too long to be worth giving.*’ We could not find any statement of the result, nor the (long) proof alluded to, in any of his later papers. Neither could we elsewhere in the literature. In this paper we shall prove some consequences of the Schur-like property that are relevant to further work on equicontinuous families of Markov operators (Section 4.1) in particular, and to coincidence of various weak and norm topologies on particular sets of signed measures (Section 4.3), among others. We stress that the main result is about *signed* measures. The similar result for positive measures appeared e.g. in [7].

In Section 3 we provide our proof of the main theorem, which is indeed also quite lengthy – and delicate. We also provide a proof for the established case of positive measures, to introduce our approach. We use a kind of geometric argument that is inspired by the work of Szarek (see [17, 18]). The observation made in Lemma 2.3 is key to this approach.

In Section 4 we provide various consequences of our main result. In Section 4.1 we prove the equivalence of concepts of equicontinuity in the theory of Markov operators and semigroups in Probability Theory in Theorem 4.1. Equicontinuous families of Markov operators were introduced in relation to asymptotic stability: the convergence of the law of stochastic Markov process to an invariant measure (e.g. e-chains [22], e-property [17, 18, 25], Cesaro-e-property [28], Ch.7; see also [15]). Hairer & Mattingly introduced the so-called asymptotic strong Feller property for that purpose [10]. Theorem 4.1 rigorously connects two views: Markov operators acting on measures (laws) and Markov operators acting on functions (observables).

Section 4.2 provides an alternative proof – for Polish spaces – of the well-known fact that the space of finite signed Borel measures is $\sigma(\mathcal{M}(S), C_b(S))$ -weakly sequentially complete, based upon Theorem 3.2.

In Section 4.3 the main Theorem 4.3 entails that the weak and norm topology in $\mathcal{M}(S)_{\text{BL}}$ coincide on sets of measures that are uniformly bounded for the total variation norm, when the relative weak topology on this set is first countable. One such case is provided when the set is uniformly tight, the other by a sphere of signed measures for the total variation norm, which is not uniformly tight. In [24], Theorem 5.38 and Corollary 5.39 come close to Theorem 4.3. A technical condition seems to prevent deriving our new result on coincidence of topologies from the results in [24].

2. PRELIMINARIES

We start with some preliminary results on Lipschitz functions on a metric space (S, d) . We denote the vector space of all real-valued Lipschitz functions by $\text{Lip}(S)$. The Lipschitz constant of $f \in \text{Lip}(S)$ is

$$|f|_L := \sup \left\{ \frac{|f(x) - f(y)|}{d(x, y)} : x, y \in S, x \neq y \right\}.$$

$\text{BL}(S)$ is the subspace of bounded functions in $\text{Lip}(S)$. It is a Banach space when equipped with the bounded Lipschitz or Dudley norm

$$\|f\|_{\text{BL}} := \|f\|_{\infty} + |f|_L.$$

The norm $\|f\|_{\text{FM}} := \max(\|f\|_{\infty}, |f|_L)$ is equivalent. $\text{BL}(S)$ is partially ordered by pointwise ordering.

Lemma 2.1. *Let $A \subset \text{BL}(S)$ be such that $\sup_{f \in A} \|f\|_{\text{BL}} < \infty$. Then $\sup(A)$ exists in $\text{BL}(S)$ and $|\sup(A)|_L \leq \sup_{f \in A} |f|_L$. In particular, $\|\sup(A)\|_{\text{BL}} \leq 2 \sup_{f \in A} \|f\|_{\text{BL}}$.*

Proof. Put $L := \sup_{f \in A} |f|_L$ and let $g = \sup(A)$, i.e. $g(x) := \sup\{f(x) : f \in A\}$ for every $x \in S$. Let $x, y \in S$. We may assume $g(x) \geq g(y)$. Let $\varepsilon > 0$. There exists $f \in A$ such that $g(x) < f(x) + \varepsilon$. By definition $g(y) \geq f(y)$. Hence

$$|g(x) - g(y)| \leq g(x) - f(x) + f(x) - f(y) < \varepsilon + |f(x) - f(y)| \leq \varepsilon + L d(x, y).$$

Since ε is arbitrary, we obtain that $|g(x) - g(y)| \leq L d(x, y)$. Thus $g \in \text{Lip}(S)$ and $|g|_L \leq L$. Clearly, $\|g\|_{\infty} \leq \sup_{f \in A} \|f\|_{\infty} < \infty$, so $g \in \text{BL}(S)$ and $\|g\|_{\text{BL}} \leq 2 \sup_{f \in A} \|f\|_{\text{BL}}$. \square

The *support* of $f \in C(S)$, denoted by $\text{supp } f$, is the closure of the set of points where f is nonzero. Lemma 2.1 implies the following

Lemma 2.2. *Let $(f_k) \subset \text{BL}(S)$ be such that $\sup_{k \geq 1} \|f_k\|_{\text{BL}} < \infty$. Assume that their supports are pairwise disjoint. Then the series $f(x) := \sum_{k=1}^{\infty} f_k(x)$ converges pointwise and $f \in \text{BL}(S)$. In particular,*

$$(2.1) \quad \|f\|_{\infty} \leq \sup_{k \geq 1} \|f_k\|_{\infty}, \quad |f|_L \leq 2 \sup_{k \geq 1} |f_k|_L.$$

Proof. Because the sets $\text{supp } f_k$ are pairwise disjoint, $f(x) = f_k(x)$ if $x \in \text{supp } f_k$. So the positive part f^+ and negative part f^- of f satisfy $f^{\pm} = \sum_{k=1}^{\infty} f_k^{\pm}$ and it suffices to prove the result for $f \geq 0$. In that case, $f = \sup_{k \geq 1} f_k$, and the first estimate in (2.1) follows immediately. The second follows from Lemma 2.1. \square

The space $\mathcal{M}(S)$ embeds into $\text{BL}(S)^*$ by means of integration: $\mu \mapsto I_{\mu}$, where

$$I_{\mu}(f) = \langle \mu, f \rangle := \int_S f d\mu.$$

The norms on $\text{BL}(S)^*$ dual to either $\|\cdot\|_{\text{BL}}$ or $\|\cdot\|_{\text{FM}}$ introduce equivalent norms on $\mathcal{M}(S)$ through the map $\mu \mapsto I_{\mu}$. These are called the bounded Lipschitz norm, or Dudley norm, and Fortet-Mourier norm on $\mathcal{M}(S)$, respectively. $\mathcal{M}(S)$ equipped with the norm topology induced by either of these norms is denoted by $\mathcal{M}(S)_{\text{BL}}$. It is not complete generally. We write $\|\cdot\|_{\text{TV}}$ for the total variation norm on $\mathcal{M}(S)$:

$$\|\mu\|_{\text{TV}} = |\mu|(S) = \mu^+(S) + \mu^-(S),$$

where $\mu = \mu^+ - \mu^-$ is the Jordan decomposition of μ . $\mathcal{M}^+(S)$ is the convex cone of positive measures in $\mathcal{M}(S)$. One has

$$(2.2) \quad \|\mu\|_{\text{TV}} = \|\mu\|_{\text{BL}}^* = \|\mu\|_{\text{FM}}^* \quad \text{for all } \mu \in \mathcal{M}^+(S).$$

In general, for $\mu \in \mathcal{M}(S)$, $\|\mu\|_{\text{BL}}^* \leq \|\mu\|_{\text{FM}}^* \leq \|\mu\|_{\text{TV}}$.

A finite signed Borel measure μ is *tight* if for every $\varepsilon > 0$ there exists a compact set $K_\varepsilon \subset S$ such that $|\mu|(S \setminus K_\varepsilon) < \varepsilon$. A family $M \subset \mathcal{M}(S)$ is *tight* or *uniformly tight* if for every $\varepsilon > 0$ there exists a compact set $K_\varepsilon \subset S$ such that $|\mu|(S \setminus K_\varepsilon) < \varepsilon$ for all $\mu \in M$. According to Prokhorov's Theorem (see [5], Theorem 8.6.2), if (S, d) is a complete separable metric space, a set of Borel probability measures $M \subset \mathcal{P}(S)$ is tight if and only if it is precompact in $\mathcal{P}(S)_{\text{BL}}$. Completeness of S is an essential condition for this theorem to hold.

Lemma 2.3. *Let (S, d) be a complete separable metric space. Let $\mu_n \in \mathcal{M}^+(S)$, $n \in \mathbb{N}$. Assume that $\{\mu_n : n \geq 1\}$ is not tight. Then there exists $\varepsilon > 0$, an increasing sequence (n_k) of positive integers and a sequence of compact sets (K_{n_k}) such that*

$$\mu_{n_k}(K_{n_k}) \geq \varepsilon \quad \text{for all } k \geq 1$$

and

$$\text{dist}(K_{n_k}, K_{n_m}) := \min\{d(x, y) \mid x \in K_{n_k}, y \in K_{n_m}\} > \varepsilon \quad \text{for all } k \neq m.$$

This result was originally stated in [17], Lemma 1, p. 1410, for a sequence (μ_n) of probability Borel measures with a proof in [18] (proof of Theorem 3.1, p. 517-518), but it is also valid for (positive) measures.

In a metric space (S, d) , if $A \subset S$ is nonempty, we write

$$A^\varepsilon := \{x \in S : d(x, A) \leq \varepsilon\}$$

for the closed ε -neighbourhood of A .

3. A SCHUR-LIKE PROPERTY FOR THE SPACE OF MEASURES

Properties of the space of Borel probability measures on S for the weak topology induced by pairing with $C_b(S)$ have been widely studied in probability theory, e.g. consult [5] for an overview. Dudley [7] studied the pairing between *signed measures* and $\text{BL}(S)$ in further detail. Results on the weak topology on signed measures induced by this pairing are the major concern of this section.

In [7] Theorem 9 the following result was obtained, rephrased in our terminology:

Theorem 3.1. *Let (S, d) be a complete separable metric space. Let $(\mu_n) \subset \mathcal{M}^+(S)$ be such that for every $f \in \text{BL}(S)$, $\langle \mu_n, f \rangle$ converges. Then $\langle \mu_n, f \rangle$ converges for every $f \in C_b(S)$. In particular, there exists $\mu \in \mathcal{M}^+(S)$ such that $\|\mu_n - \mu\|_{\text{BL}}^* \rightarrow 0$.*

Our main result, Theorem 3.2 below, generalizes this ‘*weak-implies-strong-convergence*’ property for positive measures to sequences of signed measures that are bounded in total variation norm. Its proof builds on Theorem 3.1, but not in a straightforward manner. In fact, if for a sequence (μ_n) of signed measures $\langle \mu_n, f \rangle$ is convergent for every $f \in \text{BL}(S)$, then it need not hold that $\langle \mu_n^+, f \rangle$ and $\langle \mu_n^-, f \rangle$ converge for every $f \in \text{BL}(S)$. Take for example on $S = \mathbb{R}$ with the usual Euclidean metric $\mu_n := \delta_n - \delta_{n+\frac{1}{n}}$. Then $\langle \mu_n, f \rangle \rightarrow 0$ for every $f \in \text{BL}(\mathbb{R})$. However, $\mu_n^+ = \delta_n$ and $\mu_n^- = \delta_{n+\frac{1}{n}}$, so $\langle \mu_n^\pm, f \rangle$ will not converge for every $f \in \text{BL}(\mathbb{R})$. Thus, an immediate reduction to positive measures is not possible.

In order to give a self-contained proof of the main result we include a complete proof of Theorem 3.1. It will also aid the reader in getting introduced to the type of argument we employ, based on Lemma 2.3. In addition to this lemma, the following observation is made:

Lemma 3.1. *Let $(\mu_n) \subset \mathcal{M}^+(S)$ be such that $\sup_n \mu_n(S) < \infty$ and let (E_n) be a sequence of pairwise disjoint Borel measurable subsets of S . Then for every $\varepsilon > 0$ there exists a strictly increasing subsequence (n_i) of \mathbb{N} such that for every $i \geq 1$,*

$$(3.1) \quad \mu_{n_i} \left(\bigcup_{j \neq i} E_{n_j} \right) < \varepsilon.$$

Proof. Let us first prove that for every $\eta > 0$ there exists a strictly increasing subsequence (m_i) such that

$$(3.2) \quad \mu_{m_1} \left(\bigcup_{i>1} E_{m_i} \right) < \eta$$

and

$$(3.3) \quad \mu_{m_i}(E_{m_1}) < \eta \quad \text{for all } i \geq 2.$$

Fix $\eta > 0$. Set $C := \sup_n \mu_n(S)$ and let $N \geq 1$ be such that $N\eta > C$. Since for every $n \geq 1$ we have $\sum_{m=1}^N \mu_n(E_m) = \mu_n \left(\bigcup_{m=1}^N E_m \right) \leq \mu_n(S) \leq C < N\eta$, there exists $m \in \{1, \dots, N\}$ such that

$$(3.4) \quad \mu_n(E_m) < \eta.$$

Thus there exists $m_1 \in \{1, \dots, N\}$ and an infinite set \mathcal{S} such that condition (3.4) holds for all $n \in \mathcal{S}$. Let us split \mathcal{S} into N disjoint infinite subsets $\mathcal{S}_1, \dots, \mathcal{S}_N$.

Since

$$\bigcup_{n \in \mathcal{S}_i} E_n \cap \bigcup_{n \in \mathcal{S}_j} E_n = \emptyset \quad \text{for } i, j \in \{1, \dots, N\}, i \neq j,$$

we have

$$\sum_{i=1}^N \mu_{m_1} \left(\bigcup_{n \in \mathcal{S}_i} E_n \right) = \mu_{m_1} \left(\bigcup_{i=1}^N \bigcup_{n \in \mathcal{S}_i} E_n \right) = \mu_{m_1} \left(\bigcup_{n \in \mathcal{S}} E_n \right) \leq \mu_{m_1}(S) \leq C < N\eta,$$

which, in turn, yields

$$\mu_{m_1} \left(\bigcup_{n \in \mathcal{S}_p} E_n \right) < \eta$$

for some $p \in \{1, \dots, N\}$. Now let m_2, m_3, \dots be an increasing sequence of elements from the set \mathcal{S}_p .

By induction we shall define the sequences (m_i^k) for $k \geq 1$ in the following way. First set $m_i^1 = m_i$ for $i = 1, 2, \dots$, where (m_i) is an increasing sequence satisfying conditions (3.2) and (3.3) with $\eta = \varepsilon/2$. Now if (m_i^{k-1}) is given, by what we have already proven, we may find its subsequence (m_i^k) , $m_1^k > m_1^{k-1}$, satisfying conditions (3.2) and (3.3) with $\eta = \varepsilon/2^k$.

Now set $n_i := m_1^i$ for $i = 1, 2, \dots$ and observe that

$$\mu_{n_i} \left(\bigcup_{j \neq i} E_{n_j} \right) = \sum_{j < i} \mu_{n_i}(E_{n_j}) + \mu_{n_i} \left(\bigcup_{j > i} E_{n_j} \right) \leq \sum_{j < i} \varepsilon/2^j + \varepsilon/2^i < \varepsilon.$$

The first term evaluation follows from (3.3), by the fact that n_i is an element of the sequences (m_n^j) for $j < i$. Similarly, the second term is evaluated by inequality (3.2). \square

We can now prove Theorem 3.1.

Proof. (Theorem 3.1). Let $(\mu_n) \subset \mathcal{M}^+(S)$. At the beginning we show that it is enough to prove the claim for $(\mu_n) \subset \mathcal{P}(S)$. In fact, from the assumption that $\lim_{n \rightarrow \infty} \langle \mu_n, f \rangle$ exists for every $f \in \text{BL}(S)$, in particular for $f \equiv 1$, we obtain that $\lim_{n \rightarrow \infty} \mu_n(S)$ also exists. Set $c = \lim_{n \rightarrow \infty} \mu_n(S)$ and observe that $c < \infty$, by the fact that $\sup_{n \geq 1} \|\mu_n\|_{TV} < \infty$. If $c = 0$, then we immediately see that $\mu \equiv 0$ fulfills the requirements of our theorem. On the other hand, if $c > 0$, then, we can replace μ_n with $\tilde{\mu}_n := \mu_n / \mu_n(S)$, which is a probability measure. If the theorem is proven to hold for $(\tilde{\mu}_n)$, then it holds for the (μ_n) as well.

To prove the theorem it suffices to show that the family $\{\tilde{\mu}_n : n \geq 1\}$ is tight, by the following argument. By Prokhorov's Theorem there exists some measure $\mu_* \in \mathcal{P}(S)$ and a subsequence (n_m) such that $\tilde{\mu}_{n_m} \rightarrow \mu_*$ weakly. Further, due to the fact that $\lim_{n \rightarrow \infty} \langle \tilde{\mu}_n, f \rangle$ exists for any $f \in \text{BL}(S)$, we obtain that $\lim_{n \rightarrow \infty} \langle \tilde{\mu}_n, f \rangle = \langle \mu_*, f \rangle$ for $f \in \text{BL}(S)$. This in turn, together with the tightness of $\{\tilde{\mu}_n : n \geq 1\}$, implies that $\tilde{\mu}_n \rightarrow \mu_*$ $C_b(S)$ -weakly, as $n \rightarrow \infty$. Indeed, the tightness allows restricting (approximately) to a compact subset K . The continuous bounded function on S , when restricted to K can be approximated uniformly by a function in $\text{BL}(K)$, since $\text{BL}(K) \subset C(K)$ is $\|\cdot\|_\infty$ -dense. The Metric Tietze Extension Theorem (cf. [21]) allows to extend the function in $\text{BL}(K)$ to one in $\text{BL}(S)$ without changing uniform norm and Lipschitz constant. The claim then follows. The C_b -weak convergence of $\tilde{\mu}_n$ to μ_* is equivalent to $\|\tilde{\mu}_n - \mu_*\|_{\text{BL}}^* \rightarrow 0$, as $n \rightarrow \infty$, because the latter norm metrises C_b -weak convergence on $\mathcal{M}^+(S)$ (cf. [7], Theorem 6 and Theorem 8). For $\mu = c\mu_*$ we obtain that $\|\mu_n - \mu\|_{\text{BL}}^* \rightarrow 0$, as $n \rightarrow \infty$.

To complete the proof, we have to prove the claim that the family $\{\mu_n : n \geq 1\} \subset \mathcal{P}(S)$ is uniformly tight. Assume, contrary to our claim, that it is not tight. By Lemma 2.3, passing to a subsequence if necessary, we may assume that there exists $\varepsilon > 0$ and a sequence of compact sets (K_n) satisfying

$$(3.5) \quad \mu_n(K_n) \geq \varepsilon \quad \text{for every } n \geq 1$$

and

$$(3.6) \quad \text{dist}(K_n, K_m) := \min\{\rho(x, y) : x \in K_n \text{ and } y \in K_m\} > \varepsilon \quad \text{for } m \neq n.$$

From Lemma 3.1, with $E_n := K_n^{\varepsilon/3}$, it follows that there exists a subsequence (n_i) such that for every $i \geq 1$ we have

$$(3.7) \quad \mu_{n_i} \left(\bigcup_{j \neq i} K_{n_j}^{\varepsilon/3} \right) < \varepsilon/2.$$

Note that $\text{dist}(K_{n_i}^{\varepsilon/3}, K_{n_j}^{\varepsilon/3}) > \varepsilon/3$ for $i \neq j$.

We define the function $f : X \rightarrow [0, 1]$ by the formula

$$f(x) = \sum_{i=1}^{\infty} f_i(x),$$

where f_i are arbitrary Lipschitz functions with Lipschitz constant $3/\varepsilon$ satisfying

$$f_i|_{K_{n_{2i}}} = 1 \quad \text{and} \quad 0 \leq f_i \leq \mathbf{1}_{K_{n_{2i}}^{\varepsilon/3}}.$$

According to Lemma 2.2, $f \in \text{BL}(S)$ (with $\|f\|_{\infty} \leq 1$ and $|f|_L \leq 6/\varepsilon$).

To finish the proof it is enough to observe that for every $i \geq 1$ we have

$$\langle \mu_{n_{2i}}, f \rangle = \sum_{j=1}^{\infty} \langle \mu_{n_{2i}}, f_j \rangle \geq \mu_{n_{2i}}(K_{n_{2i}}) \stackrel{(3.5)}{\geq} \varepsilon$$

and

$$\langle \mu_{n_{2i+1}}, f \rangle = \sum_{j=1}^{\infty} \langle \mu_{n_{2i+1}}, f_j \rangle \leq \sum_{j=1}^{\infty} \mu_{n_{2i+1}}(K_{n_{2j}}^{\varepsilon/3}) \leq \mu_{n_{2i+1}} \left(\bigcup_{j \neq 2i+1} K_{n_j}^{\varepsilon/3} \right) \stackrel{(3.7)}{<} \varepsilon/2,$$

which contradicts the assumption that $\lim_{n \rightarrow \infty} \langle \mu_n, f \rangle$ exists for every $f \in \text{BL}(S)$. Thus the family $\{\mu_n : n \geq 1\}$ is tight and we are done. \square

Remark 3.1. In the proof we show that if (μ_n) is a sequence of positive Borel measures such that $\langle \mu_n, f \rangle$ converges for every $f \in \text{BL}(S)$, then (μ_n) is uniformly tight in $\mathcal{M}^+(S)$. See [5], Corollary 8.6.3, p. 204, for results in this direction when $\langle \mu_n, f \rangle$ converges for every $f \in C_b(S)$. Under the additional condition that there exists $\mu_* \in \mathcal{M}^+(S)$ such that $\langle \mu_n, f \rangle \rightarrow \langle \mu_*, f \rangle$ for every $f \in C_b(S)$, tightness results appeared already in e.g. [20], Theorem 4 for positive measures or [4], Appendix III, Theorem 8 for probability measures.

We shall now prove the main result of this paper which we restate for convenience of the reader:

Theorem 3.2. *Let (S, d) be a complete separable metric space. Let $(\mu_n) \subset \mathcal{M}(S)$ be such that $\sup_n \|\mu_n\|_{TV} < \infty$. If for every $f \in \text{BL}(S)$ the sequence $\langle \mu_n, f \rangle$ converges, then there exists $\mu \in \mathcal{M}(S)$ such that $\|\mu_n - \mu\|_{BL}^* \rightarrow 0$ as $n \rightarrow \infty$.*

Proof. Let $(\mu_n) \subset \mathcal{M}(S)$ be signed measures such that $\sup_n \|\mu_n\|_{TV} < \infty$. Denote by μ_n^+ and μ_n^- the positive and negative part of μ_n , $n \geq 1$, respectively. We consider the following set

$$\begin{aligned} \mathcal{C} := \Big\{ (\beta, (m_n), (\nu_{m_n}), (\vartheta_{m_n})) : \beta \geq 0, (m_n) \subset \mathbb{N} \text{ -- an increasing sequence,} \\ \nu_{m_n}, \vartheta_{m_n} \in \mathcal{P}(S), \lim_{n \rightarrow \infty} \|\nu_{m_n} - \vartheta_{m_n}\|_{BL}^* = 0 \\ \text{and } \mu_{m_n}^+ \geq \beta \nu_{m_n}, \mu_{m_n}^- \geq \beta \vartheta_{m_n} \Big\}. \end{aligned}$$

We first observe that $\mathcal{C} \neq \emptyset$, which follows from the fact that $(0, (m_n), (\nu_{m_n}), (\vartheta_{m_n})) \in \mathcal{C}$ for arbitrary (m_n) and $\nu_{m_n}, \vartheta_{m_n} \in \mathcal{P}(S)$ such that $\lim_{n \rightarrow \infty} \|\nu_{m_n} - \vartheta_{m_n}\|_{BL}^* = 0$. Moreover, since $\bar{c} := \sup_{n \geq 1} \|\mu_n\|_{TV} < \infty$, we obtain that $0 \leq \beta \leq \bar{c}$ for every β for which there are some (m_n) and $\nu_{m_n}, \vartheta_{m_n}$ such that $(\beta, (m_n), (\nu_{m_n}), (\vartheta_{m_n})) \in \mathcal{C}$. We can therefore introduce

$$\alpha = \sup \{ \beta : (\beta, (m_n), (\nu_{m_n}), (\vartheta_{m_n})) \in \mathcal{C} \}.$$

From the definition of α it follows that there exists a subsequence (m_n) of positive integers and an increasing sequence (α_n) of nonnegative constants satisfying $\lim_{n \rightarrow \infty} \alpha_n = \alpha$ and

$$\mu_{m_n}^+ \geq \alpha_n \nu_{m_n} \quad \text{and} \quad \mu_{m_n}^- \geq \alpha_n \vartheta_{m_n},$$

where $\nu_{m_n}, \vartheta_{m_n} \in \mathcal{P}(S)$ are such that $\|\nu_{m_n} - \vartheta_{m_n}\|_{BL}^* \rightarrow 0$ as $n \rightarrow \infty$.

To finish the proof it is enough to show that both the sequences $(\mu_{m_n}^+ - \alpha_n \nu_{m_n})$ and $(\mu_{m_n}^- - \alpha_n \vartheta_{m_n})$ are tight. Indeed, then, by the Prokhorov Theorem ([5], Theorem 8.6.2) there exists a subsequence (m_{n_k}) of (m_n) and two measures μ^1 and μ^2 such that the sequences $(\mu_{m_{n_k}}^+ - \alpha_{n_k} \nu_{m_{n_k}})$ and $(\mu_{m_{n_k}}^- - \alpha_{n_k} \vartheta_{m_{n_k}})$ converge $C_b(S)$ -weakly to a positive measure μ^1 and μ^2 , respectively. Hence also in $\|\cdot\|_{BL}^*$ -norm, according to Theorem 3.1. Consequently, $\|\mu_{m_{n_k}} - (\mu^1 - \mu^2)\|_{BL}^* \rightarrow 0$ as $k \rightarrow \infty$, by the fact that $\|\nu_{m_{n_k}} - \vartheta_{m_{n_k}}\|_{BL}^* \rightarrow 0$ as $k \rightarrow \infty$. This will complete the proof of the theorem. Indeed, if we know that the sequence (and also any subsequence) has a convergent subsequence (in the dual bounded Lipschitz norm), then the sequence is also convergent due to the fact that the limit of all convergent subsequences is the same, by the assumption that $\lim_{n \rightarrow \infty} \langle \mu_n, f \rangle$ exists for any $f \in \text{BL}(S)$.

Assume now, contrary to our claim, that at least one of the families $(\mu_{m_n}^+ - \alpha_n \nu_{m_n})$ or $(\mu_{m_n}^- - \alpha_n \vartheta_{m_n})$, say the first one, is not tight. By Lemma 2.3, passing to a subsequence

if necessary, we may assume that there exists $\varepsilon > 0$ and a sequence of compact sets (K_n) satisfying

$$(3.8) \quad (\mu_{m_n}^+ - \alpha_n \nu_{m_n})(K_n) \geq \varepsilon$$

and

$$\text{dist}(K_i, K_j) \geq \varepsilon \quad \text{for } i, j \in \mathbb{N}, i \neq j.$$

Set

$$\tilde{\mu}_n := \mu_{m_n}^+ - \alpha_n \nu_{m_n} \quad \text{and} \quad \hat{\mu}_n := \mu_{m_n}^- - \alpha_n \vartheta_{m_n}.$$

Claim: For any $0 < \eta \leq 1$ there exist j , as large as we wish, and $\tau_j, \chi_j \in \mathcal{P}(S)$ satisfying

$$\tilde{\mu}_j \geq (\varepsilon/2)\tau_j, \quad \hat{\mu}_j \geq (\varepsilon/2)\chi_j \quad \text{and} \quad \|\tau_j - \chi_j\|_{\text{BL}}^* \leq \eta.$$

Consequently, there will exist a subsequence (m_{j_n}) such that

$$\mu_{m_{j_n}}^+ = \alpha_{j_n} \nu_{m_{j_n}} + \tilde{\mu}_{j_n} \geq \alpha_{j_n} \nu_{m_{j_n}} + (\varepsilon/2)\tau_{j_n},$$

$$\mu_{m_{j_n}}^- \geq \alpha_{j_n} \vartheta_{m_{j_n}} + (\varepsilon/2)\chi_{j_n} \quad \text{and} \quad \|\tau_{j_n} - \chi_{j_n}\|_{\text{BL}}^* \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Now, if we define probability measures $\varrho_{m_{j_n}}, \varsigma_{m_{j_n}}$ as follows

$$\varrho_{m_{j_n}} := (\alpha_{j_n} \nu_{m_{j_n}} + (\varepsilon/2)\tau_{j_n})(\alpha_{j_n} + \varepsilon/2)^{-1}, \quad \varsigma_{m_{j_n}} := (\alpha_{j_n} \vartheta_{m_{j_n}} + (\varepsilon/2)\chi_{j_n})(\alpha_{j_n} + \varepsilon/2)^{-1},$$

we will obtain

$$\mu_{m_{j_n}}^+ \geq (\alpha_{j_n} + \varepsilon/2)\varrho_{m_{j_n}}, \quad \mu_{m_{j_n}}^- \geq (\alpha_{j_n} + \varepsilon/2)\varsigma_{m_{j_n}}$$

and $\lim_{n \rightarrow \infty} \|\varrho_{m_{j_n}} - \varsigma_{m_{j_n}}\|_{\text{BL}}^* = 0$, which is impossible, because it contradicts the definition of α , since $\lim_{n \rightarrow \infty} (\alpha_{j_n} + \varepsilon/2) > \alpha$.

Let us prove the claim. Set $\xi_n := \tilde{\mu}_n + \hat{\mu}_n$ for $n \geq 1$ and let $C := \sup_{n \geq 1} \xi_n(S)$. Observe that $C \leq \sup_{n \geq 1} \|\mu_n\|_{\text{TV}} < \infty$. Fix $0 < \eta \leq 1$ and let $\kappa \in (0, \varepsilon/6)$ be such that $6\kappa(1/\varepsilon + 2/\varepsilon^2) < \eta$. Lemma 3.1 yields an increasing sequence $(j_n) \subset \mathbb{N}$ such that

$$(3.9) \quad \xi_{j_n} \left(\bigcup_{l \neq n} K_{j_l}^{\varepsilon/3} \right) < \kappa/4$$

and hence

$$\tilde{\mu}_{j_n} \left(\bigcup_{l \neq n} K_{j_l}^{\varepsilon/3} \right) < \kappa/4 \quad \text{and} \quad \hat{\mu}_{j_n} \left(\bigcup_{l \neq n} K_{j_l}^{\varepsilon/3} \right) < \kappa/4$$

for all $n = 1, 2, \dots$

Choose $N \geq 1$ such that $N\kappa/4 > C$ and set $W_{j_n}^p := K_{j_n}^{p\varepsilon/(3N)} \setminus K_{j_n}^{(p-1)\varepsilon/(3N)}$ for $p = 1, \dots, N$. Observe that $W_{j_n}^p \cap W_{j_n}^q = \emptyset$ for $p \neq q$. Since $\sum_{p=1}^N \xi_{j_n}(W_{j_n}^p) = \xi_{j_n}(\bigcup_{p=1}^N W_{j_n}^p) \leq C$, $n \geq 1$, for every n there exists $p_n \in \{1, \dots, N\}$ such that

$$(3.10) \quad \xi_{j_n}(W_{j_n}^{p_n}) < \kappa/4.$$

Now we are in a position to define a sequence (f_n) of functions from S to $[-1, 1]$. The construction is as follows. For $n = 2k + 1$ for $k \geq 1$, we set $f_n \equiv 0$. On the other hand, to define functions f_n for $n = 2k$ we introduce the measures

$$\tilde{\mu}'_{j_n}(\cdot) = \tilde{\mu}_{j_n} \left(\cdot \cap K_{j_n}^{(p_n-1)\varepsilon/(3N)} \right)$$

and

$$\hat{\mu}'_{j_n}(\cdot) = \hat{\mu}_{j_n} \left(\cdot \cap K_{j_n}^{(p_n-1)\varepsilon/(3N)} \right).$$

Further, there exists a Lipschitz function $\tilde{f}_n : K_{j_n}^{(p_n-1)\varepsilon/(3N)} \rightarrow [-1, 1]$ with $|\tilde{f}_n|_L \leq 1$ such that $\langle \tilde{\mu}'_{j_n} - \hat{\mu}'_{j_n}, \tilde{f}_n \rangle \geq \frac{1}{2} \|\tilde{\mu}'_{j_n} - \hat{\mu}'_{j_n}\|_{\text{BL}}^*$. Let f_n be a Lipschitz extension of the function \tilde{f}_n to S such that $f_n(x) = \tilde{f}_n(x)$ for $x \in K_{j_n}^{(p_n-1)\varepsilon/(3N)}$ and $f_n(x) = 0$ for $x \notin K_{j_n}^{p_n\varepsilon/(3N)}$. We may assume that $|f_n|_L \leq 3N/\varepsilon$. The existence of the extension function follows from McShane's formula (see [21]). Let $f = \sum_{k=1}^{\infty} f_{2k}$. Since $\text{dist}(\text{supp } f_i, \text{supp } f_j) > \varepsilon/3$ for $i, j \geq 1, i \neq j$, f is a bounded Lipschitz function, by Lemma 2.2.

We show that $\langle \mu_{m_{j_i}}, f \rangle \leq \kappa/2$ for $i = 2k + 1$. Indeed, for k sufficiently large we have

$$\begin{aligned} \langle \mu_{m_{j_{2k+1}}}, f \rangle &= \sum_{n=1}^{\infty} \langle \mu_{m_{j_{2k+1}}}, f_{2n} \rangle \leq \sum_{n=1}^{\infty} \xi_{j_{2k+1}} \left(K_{j_{2n}}^{\varepsilon/3} \right) + \alpha_{j_{2k+1}} \|\nu_{m_{j_{2k+1}}} - \vartheta_{m_{j_{2k+1}}}\|_{\text{BL}}^* \\ &\leq \xi_{j_{2k+1}} \left(\bigcup_{l \neq 2k+1} K_{j_l}^{\varepsilon/3} \right) + \alpha_{j_{2k+1}} \|\nu_{m_{j_{2k+1}}} - \vartheta_{m_{j_{2k+1}}}\|_{\text{BL}}^* \\ (3.9) \quad &< \kappa/4 + \alpha_{j_{2k+1}} \|\nu_{m_{j_{2k+1}}} - \vartheta_{m_{j_{2k+1}}}\|_{\text{BL}}^* < \kappa/2, \end{aligned}$$

by the properties of the measures $\nu_{m_{j_{2k+1}}}, \vartheta_{m_{j_{2k+1}}}$ and the definition of the functions f_{2n} . Therefore

$$\lim_{i \rightarrow \infty} \langle \mu_{m_{j_i}}, f \rangle = \lim_{k \rightarrow \infty} \langle \mu_{m_{j_{2k+1}}}, f \rangle \leq \kappa/2,$$

because we assume that the limit of $\langle \mu_m, f \rangle$ exists.

On the other hand, for $i = 2k$ we have

$$\begin{aligned} \langle \mu_{m_{j_{2k}}}, f \rangle &= \sum_{n=1}^{\infty} \langle \mu_{m_{j_{2k}}}, f_{2n} \rangle \geq - \sum_{n \neq k}^{\infty} \xi_{j_{2k}} \left(K_{j_{2n}}^{\varepsilon/3} \right) + \langle \mu_{m_{j_{2k}}}, f_{2n} \rangle \\ &\geq - \sum_{n \neq k}^{\infty} \xi_{j_{2k}} \left(K_{j_{2n}}^{\varepsilon/3} \right) - \xi_{j_{2k}} \left(W_{j_{2k}}^{p_{2k}} \right) + \langle \tilde{\mu}'_{j_{2k}} - \hat{\mu}'_{j_{2k}}, \tilde{f}_{2k} \rangle \\ &\geq -\kappa/4 - \kappa/4 + \frac{1}{2} \|\tilde{\mu}'_{j_{2k}} - \hat{\mu}'_{j_{2k}}\|_{\text{BL}}^*, \end{aligned}$$

by the fact that $\|f_{2n}\|_\infty \leq 1$. Since $\lim_{i \rightarrow \infty} \langle \mu_{m_{j_i}}, f \rangle \leq \kappa/2$, by the estimation obtained for $i = 2k + 1$ and the assumption that the limit exists, we have

$$-\kappa/4 - \kappa/4 + \frac{1}{2} \|\tilde{\mu}'_{j_{2k}} - \hat{\mu}'_{j_{2k}}\|_{\text{BL}}^* \leq 3\kappa/4$$

for k sufficiently large and consequently

$$\|\tilde{\mu}'_{j_{2k}} - \hat{\mu}'_{j_{2k}}\|_{\text{BL}}^* \leq 3\kappa$$

for all k sufficiently large. Thus

$$\hat{\mu}'_{j_{2k}}(S) \geq \tilde{\mu}'_{j_{2k}}(S) - 3\kappa \geq \varepsilon - \varepsilon/2 = \varepsilon/2.$$

Hence, for probability measures

$$\tilde{\nu}_{j_{2k}} := \tilde{\mu}'_{j_{2k}} / \tilde{\mu}'_{j_{2k}}(S) \quad \text{and} \quad \hat{\nu}_{j_{2k}} := \hat{\mu}'_{j_{2k}} / \hat{\mu}'_{j_{2k}}(S)$$

we have for k sufficiently large

$$\tilde{\mu}_{j_{2k}} \geq \tilde{\mu}'_{j_{2k}} \geq (\varepsilon/2)\tilde{\nu}_{j_{2k}} \quad \text{and} \quad \hat{\mu}_{j_{2k}} \geq \hat{\mu}'_{j_{2k}} \geq (\varepsilon/2)\hat{\nu}_{j_{2k}}.$$

Finally, observe that for k sufficiently large,

$$\begin{aligned} \|\tilde{\nu}_{j_{2k}} - \hat{\nu}_{j_{2k}}\|_{\text{BL}}^* &\leq \|\tilde{\mu}'_{j_{2k}} / \tilde{\mu}'_{j_{2k}}(S) - \hat{\mu}'_{j_{2k}} / \hat{\mu}'_{j_{2k}}(S)\|_{\text{BL}}^* + \|\hat{\mu}'_{j_{2k}}\|_{\text{BL}}^* |1/\tilde{\mu}'_{j_{2k}}(S) - 1/\hat{\mu}'_{j_{2k}}(S)| \\ &\leq (1/\tilde{\mu}'_{j_{2k}}(S)) \|\tilde{\mu}'_{j_{2k}} - \hat{\mu}'_{j_{2k}}\|_{\text{BL}}^* + 1/(\tilde{\mu}'_{j_{2k}}(S)\hat{\mu}'_{j_{2k}}(S)) |\tilde{\mu}'_{j_{2k}}(S) - \hat{\mu}'_{j_{2k}}(S)| \\ &\leq 6\kappa/\varepsilon + 12\kappa/\varepsilon^2 < \eta, \end{aligned}$$

by the fact that $\tilde{\mu}'_{j_{2k}}(S), \hat{\mu}'_{j_{2k}}(S) \geq \varepsilon/2$ and $|\tilde{\mu}'_{j_{2k}}(S) - \hat{\mu}'_{j_{2k}}(S)| \leq \|\tilde{\mu}'_{j_{2k}} - \hat{\mu}'_{j_{2k}}\|_{\text{BL}}^* \leq 3\kappa$. This completes the proof of the claim, hence the theorem. \square

Remark 3.2. *It is possible to prove Theorem 3.2 by means of a reduction-to- ℓ^1 -trick, inspired by ideas in [23, 24], cf. [11]. Another proof is feasible, starting from [23], Theorem 3.2, see [28]. However, here we prefer to present an independent, ‘set-geometric’ proof that is self-contained and founded on the well-established result for the case of positive measures, Theorem 3.1.*

The condition that the sequence of measures must be bounded in total variation norm cannot be omitted as the following counterexample indicates.

Counterexample. Let $S = [0, 1]$ with the Euclidean metric. Let $d\mu_n := n \sin(2\pi nx) dx$, where dx is Lebesgue measure on S . Then $\|\mu_n\|_{\text{TV}}$ is unbounded. Let $g \in \text{BL}(S)$ with $|g|_L \leq 1$. According to Rademacher’s Theorem, g is differentiable Lebesgue almost everywhere. Since $|g|_L \leq 1$, there exists $f \in L^\infty([0, 1])$ such that for all $0 \leq a < b \leq 1$,

$$\int_a^b f(x) dx = g(b) - g(a).$$

This yields

$$\langle \mu_n, g \rangle = \frac{1}{2\pi} \int_0^1 \cos(2\pi nx) f(x) dx.$$

Since $f \in L^2([0, 1])$, it follows from Bessel's Inequality that

$$\lim_{n \rightarrow \infty} \int_0^1 \cos(2\pi n x) f(x) dx = 0.$$

So $\langle \mu_n, g \rangle \rightarrow 0$ for all $g \in \text{BL}(S)$. Now, let $g_n \in \text{BL}(S)$ be the piecewise linear function that satisfies $g_n(0) = 0 = g_n(1)$,

$$g_n\left(\frac{1+4i}{4n}\right) = \frac{1}{4n}, \quad g_n\left(\frac{3+4i}{4n}\right) = -\frac{1}{4n}, \quad \text{for } i \in \mathbb{N}, 0 \leq i \leq n-1.$$

Then $|g|_L = 1$ and $\|g_n\|_\infty = \frac{1}{4n}$. An easy calculation shows that $\langle \mu_n, g_n \rangle = \frac{1}{\pi^2}$ for all $n \in \mathbb{N}$. Therefore $\|\mu_n\|_{\text{BL}}^*$ cannot converge to zero as $n \rightarrow \infty$.

4. APPLICATIONS OF THE MAIN RESULT

In this section we will illustrate the relevance of the main result by showing its applicability in various fields of mathematics that consider topologies on spaces of measures or operators on such spaces by providing some examples.

4.1. An application to equicontinuous families of Markov operators. In this section we present an intriguing consequence of Theorem 3.2 in the theory of Markov operators and semigroups.

A Markov operator on (measures on) S is a map $P : \mathcal{M}^+(S) \rightarrow \mathcal{M}^+(S)$ such that:

- (i) $P(\mu + \nu) = P\mu + P\nu$ and $P(r\mu) = rP\mu$ for all $\mu, \nu \in \mathcal{M}^+(S)$ and $r \geq 0$,
- (ii) $(P\mu)(S) = \mu(S)$ for all $\mu \in \mathcal{M}^+(S)$.

In particular, a Markov operator leaves invariant the convex set $\mathcal{P}(S)$ of probability measures in $\mathcal{M}^+(S)$.

Let $\text{BM}(S)$ be the vector space of bounded Borel measurable real-valued functions on S . A Markov operator is called *regular* if there exists a linear map $U : \text{BM}(S) \rightarrow \text{BM}(S)$, the *dual operator*, such that

$$\langle P\mu, f \rangle = \langle \mu, Uf \rangle \quad \text{for all } \mu \in \mathcal{M}^+(S), f \in \text{BM}(S).$$

A regular Markov operator P is *Feller* if its dual operator maps $C_b(S)$ into itself. Equivalently, P is continuous for the $\|\cdot\|_{\text{BL}}^*$ -norm topology (cf. e.g. [13] Lemma 3.3 and [28] Lemma 3.3.2). Let T be a topological space and (S', d') a metric space. A family of functions $\mathcal{E} \subset C(T, S')$ is *equicontinuous at* $t_0 \in T$ if for every $\varepsilon > 0$ there exists an open neighbourhood U_ε of t_0 such that

$$d'(f(t), f(t_0)) < \varepsilon \quad \text{for all } f \in \mathcal{E}, t \in U_\varepsilon.$$

\mathcal{E} is *equicontinuous* if it is equicontinuous at every point of T .

Following Szarek *et al.* [17, 25], a family $(P_\lambda)_{\lambda \in \Lambda}$ of regular Markov operators has the *e-property* if for each $f \in \text{BL}(S)$ the family $\{U_\lambda f : \lambda \in \Lambda\}$ is equicontinuous in $C_b(S)$. In

particular one may consider the family of iterates of a single Markov operator P : $(P^n)_{n \in \mathbb{N}}$, or Markov semigroups $(P_t)_{t \in \mathbb{R}^+}$, where each P_t is a regular Markov operator and $P_0 = I$, $P_t P_s = P_{t+s}$. Markov operators and semigroups with the e-property have convenient properties concerning existence, uniqueness and asymptotic stability of invariant measures, see e.g. [12, 17, 25, 26, 28]. They occur for example in the theory of Iterated Function Systems [3, 19] and stochastic differential equations [17].

Theorem 3.2 allows us to establish

Theorem 4.1. *Let $\{P_\lambda : \lambda \in \Lambda\}$ be a family of regular Markov operators on a complete metric space (S, d) . Let U_λ be the dual Markov operator of P_λ . The following statements are equivalent:*

- (i) $\{P_\lambda : \lambda \in \Lambda\}$ is equicontinuous in $C(\mathcal{M}^+(S)_{\text{BL}}, \mathcal{M}^+(S)_{\text{BL}})$,
- (ii) $\{U_\lambda f : \lambda \in \Lambda\}$ is equicontinuous in $C_b(S)$ for every $f \in \text{BL}(S)$.

Proof. (i) \Rightarrow (ii). Let $f \in \text{BL}(S)$ and $x_0 \in S$. Let $\varepsilon > 0$. Since $\{P_\lambda : \lambda \in \Lambda\}$ is equicontinuous at δ_{x_0} there exists an open neighbourhood V of δ_{x_0} in $\mathcal{M}^+(S)_{\text{BL}}$ such that

$$\|P_\lambda \delta_{x_0} - P_\lambda \mu\|_{\text{BL}}^* < \varepsilon / (1 + \|f\|_{\text{BL}}) \quad \text{for all } \lambda \in \Lambda \text{ and } \mu \in U_0.$$

Since the map $x \mapsto \delta_x : S \rightarrow \mathcal{M}^+(S)_{\text{BL}}$ is continuous, there exists an open neighbourhood V_0 of x_0 in S such that $\delta_x \in V$ for all $x \in V_0$. Then

$$|U_\lambda f(x) - U_\lambda f(x_0)| = |\langle P_\lambda \delta_x - P_\lambda \delta_{x_0}, f \rangle| \leq \frac{\varepsilon}{1 + \|f\|_{\text{BL}}} \cdot \|f\|_{\text{BL}} < \varepsilon$$

for all $x \in V_0$ and $\lambda \in \Lambda$.

(ii) \Rightarrow (i). Assume on the contrary that $\{P_\lambda : \lambda \in \Lambda\}$ is not an equicontinuous family of maps. Then there exists a point $\mu_0 \in \mathcal{M}^+(S)$ at which this family is not equicontinuous. Hence there exists $\varepsilon_0 > 0$ such that for every $k \in \mathbb{N}$ there are $\lambda_k \in \Lambda$ and $\mu_k \in \mathcal{M}^+(S)$ such that

$$(4.1) \quad \|\mu_k - \mu_0\|_{\text{BL}}^* < \frac{1}{k} \quad \text{and} \quad \|P_{\lambda_k} \mu_k - P_{\lambda_k} \mu_0\|_{\text{BL}}^* \geq \varepsilon_0 \quad \text{for all } k \in \mathbb{N}.$$

Because the measures μ_k are positive and the $\|\cdot\|_{\text{BL}}^*$ -norm metrizes the $C_b(S)$ -weak topology on $\mathcal{M}^+(S)$ (cf. [7], Theorem 18), $\langle \mu_k, f \rangle \rightarrow \langle \mu_0, f \rangle$ for every $f \in C_b(S)$. According to [7], Theorem 7, this convergence is uniform on any equicontinuous and uniformly bounded subset \mathcal{E} of $C_b(S)$. By assumption, $\mathcal{M}_f := \{U_{\lambda_k} f : k \in \mathbb{N}\}$ is such a family for every $f \in \text{BL}(S)$. Therefore

$$(4.2) \quad |\langle P_{\lambda_k} \mu_k - P_{\lambda_k} \mu_0, f \rangle| = |\langle \mu_k - \mu_0, U_{\lambda_k} f \rangle| \rightarrow 0$$

as $k \rightarrow \infty$ for every $f \in \text{BL}(S)$. Since for positive measures μ one has $\|\mu\|_{\text{TV}} = \|\mu\|_{\text{BL}}^*$, one obtains

$$|\|\mu_k\|_{\text{TV}} - \|\mu_0\|_{\text{TV}}| \leq \|\mu_k - \mu_0\|_{\text{BL}}^* \rightarrow 0.$$

So $m_0 := \sup_{k \geq 1} \|\mu_k\|_{\text{TV}} < \infty$. Moreover,

$$\|P_{\lambda_k} \mu_k - P_{\lambda_k} \mu_0\|_{\text{TV}} \leq \|P_{\lambda_k} \mu_k\|_{\text{TV}} + \|P_{\lambda_k} \mu_0\|_{\text{TV}} \leq \|\mu_k\|_{\text{TV}} + \|\mu_0\|_{\text{TV}} \leq m_0 + \|\mu_0\|_{\text{TV}}.$$

Theorem 3.2 and (4.2) yields that $\|P_{\lambda_k}\mu_k - P_{\lambda_k}\mu_0\|_{\text{BL}}^* \rightarrow 0$ as $k \rightarrow \infty$. This contradicts the second property in (4.1). \square

A particular class of examples of Markov operators and semigroups is furnished by the lift of a map or semigroup $(\phi_t)_{t \geq 0}$ of measurable maps $\phi_t : S \rightarrow S$ to measures on S by means of push-forward:

$$P_t^\phi \mu(E) := \mu(\phi_t^{-1}(E))$$

for every Borel set E of S and $\mu \in \mathcal{M}^+(S)$. A consequence of Theorem 4.1 is:

Proposition 4.1. *Let (S, d) be a complete separable metric space and let $(\phi_t)_{t \geq 0}$ be a semigroup of Borel measurable transformations of S . Then P_t^ϕ is a regular Markov operator for each $t \geq 0$. Moreover, $(P_t^\phi)_{t \geq 0}$ is equicontinuous in $C(\mathcal{M}^+(S)_{\text{BL}}, \mathcal{M}^+(S)_{\text{BL}})$ if and only if $(\phi_t)_{t \geq 0}$ is equicontinuous in $C(S, S)$.*

Proof. The regularity of P_t^ϕ is immediate, as $U_t^\phi f = f \circ \phi_t$.

‘ \Rightarrow ’: Let $x_0 \in S$ and $\varepsilon > 0$. Define $h(x) := 2x/(2+x)$ and put $\varepsilon' := h(\varepsilon)$. By equicontinuity of $(P_t^\phi)_{t \geq 0}$ at δ_{x_0} , there exists an open neighbourhood U of δ_{x_0} in $\mathcal{M}^+(S)_{\text{BL}}$ such that

$$\|P_t^\phi \mu - P_t^\phi \delta_{x_0}\|_{\text{BL}}^* < \varepsilon'$$

for all $t \geq 0$ and $\mu \in U$. Because the map $\delta : x \mapsto \delta_x : S \rightarrow \mathcal{M}^+(S)_{\text{BL}}$ is continuous, $U_0 := \delta^{-1}(U)$ is open in S . It contains x_0 . Moreover,

$$\|P_t^\phi \delta_x - P_t^\phi \delta_{x_0}\|_{\text{BL}}^* = \|\delta_{\phi_t(x)} - \delta_{\phi_t(x_0)}\|_{\text{BL}}^* = h(d(\phi_t(x), \phi_t(x_0))) < \varepsilon'$$

for all $x \in U_0$ and $t \geq 0$ (see [14] Lemma 3.5). Because h is monotone increasing,

$$d(\phi_t(x), \phi_t(x_0)) < \varepsilon \quad \text{for all } x \in U_0, t \geq 0.$$

‘ \Leftarrow ’: This part involves Theorem 4.1. Let $f \in \text{BL}(S)$. Let U_t be the dual operator of P_t . Then for all $x, x_0 \in S$,

$$|U_t f(x) - U_t f(x_0)| = |f(\phi_t(x)) - f(\phi_t(x_0))| \leq |f|_L d(\phi_t(x), \phi_t(x_0)),$$

from which the equicontinuity of $\{U_t f : t \geq 0\}$ follows. The result is obtained by applying Theorem 4.1. \square

In subsequent work we shall examine various further consequences of Theorem 4.1 for the theory and application of equicontinuous families of Markov operators. Some results in this direction were discussed in parts of [28], Chapter 7.

4.2. An alternative proof for weak sequential completeness. Theorem 3.2 allows – in the case of a Polish space – to give an alternative proof of the well-known fact that $\mathcal{M}(S)$ is $C_b(S)$ -weakly sequentially complete, that goes back to Alexandrov [2] and Varadarajan [27], see. e.g. [7], Theorem 1 or [5], Theorem 8.7.1 for a more general topological setting. We include our proof based on Theorem 3.2 here, because it employs an argument for reduction to functions in $\text{BL}(S)$, which by itself is an interesting result.

This reduction is based on the following observation. Let \mathcal{D}_S be the set of all metrics on S that metrize the topology of S as a complete separable metric space. We need to stress the dependence of the space $\text{BL}(S)$ on the chosen metric on S . So for $d \in \mathcal{D}_S$ we write $\text{BL}(S, d)$ for the space of bounded Lipschitz functions on (S, d) . The key observation is, that

$$(4.3) \quad C_b(S) = \bigcup_{d \in \mathcal{D}_S} \text{BL}(S, d).$$

In fact, fix $d_0 \in \mathcal{D}_S$. If $f \in C_b(S)$, then

$$d_f(x, y) := d_0(x, y) \vee |f(x) - f(y)|$$

is a metric on S such that $d_f \in \mathcal{D}_S$ and $f \in \text{BL}(S, d_f)$. Here \vee denotes the maximum.

The precise statement we consider is the following:

Theorem 4.2 (Weak sequential completeness). *Let S be a Polish space. Let $(\mu_n) \subset \mathcal{M}(S)$ be such that $\langle \mu_n, f \rangle$ converges for every $f \in C_b(S)$. Then there exists $\mu_* \in \mathcal{M}(S)$ such that $\langle \mu_n, f \rangle \rightarrow \langle \mu_*, f \rangle$ for every $f \in C_b(S)$.*

Proof. The norm of μ_n viewed as a continuous linear functional on $C_b(S)$ is its total variation norm. Hence, according to the Banach-Steinhaus Theorem, $\sup_{n \geq 1} \|\mu_n\|_{\text{TV}} < \infty$. For any $d \in \mathcal{D}_S$, $\langle \mu_n, f \rangle$ converges for every $f \in C_b(S)$, so in particular for every $f \in \text{BL}(S, d)$. The sequence (μ_n) is bounded in total variation norm, so Theorem 3.2 implies there exists $\mu_*^d \in \mathcal{M}(S)$ such that $\langle \mu_n, f \rangle \rightarrow \langle \mu_*^d, f \rangle$ for every $f \in \text{BL}(S, d)$. We proceed to show that the limit measure μ_*^d is independent of d .

Let $d' \in \mathcal{D}_S$. Put

$$\bar{d}(x, y) := d(x, y) \vee d'(x, y).$$

Then $\bar{d} \in \mathcal{D}_S$ and $\text{BL}(S, \bar{d})$ contains both $\text{BL}(S, d)$ and $\text{BL}(S, d')$. Let $C \subset S$ be closed. There exist sequences (h_n) and (h'_n) in $\text{BL}(S, d)$ and $\text{BL}(S, d')$ respectively, such that $h_n \downarrow \mathbb{1}_C$ and $h'_n \downarrow \mathbb{1}_C$ pointwise. Both these sequences are in $\text{BL}(S, \bar{d})$, so

$$\mu_*^d(C) = \lim_{k \rightarrow \infty} \langle \mu_*^d, h_k \rangle = \lim_{k \rightarrow \infty} \lim_{n \rightarrow \infty} \langle \mu_n, h_k \rangle = \lim_{k \rightarrow \infty} \langle \mu_*^{\bar{d}}, h_k \rangle = \mu_*^{\bar{d}}(C).$$

A similar argument applies to $\mu_*^{d'}$, using the sequence (h'_n) in $\text{BL}(S, d')$ instead of (h_n) . So μ_*^d and $\mu_*^{d'}$ (and $\mu_*^{\bar{d}}$) agree on the π -system consisting of closed sets, which generate the Borel σ -algebra. Hence these measures are equal on all Borel sets. That is, there exists $\mu_* \in \mathcal{M}(S)$ such that $\langle \mu_n, f \rangle \rightarrow \langle \mu_*, f \rangle$ for every $f \in \text{BL}(S, d)$ for every $d \in \mathcal{D}_S$. Thus for every $f \in C_b(S)$ in view of (4.3). \square

4.3. Coincidence of weak and norm topologies. A further consequence of Theorem 3.2 is

Theorem 4.3. *Let (S, d) be a complete separable metric space and let $M \subset \mathcal{M}(S)$ be such that $m := \sup_{\mu \in M} \|\mu\|_{\text{TV}} < \infty$. If the restriction of the $\sigma(\mathcal{M}(S), \text{BL}(S))$ -weak topology to M is first countable, then this topology coincides with the restriction of the $\|\cdot\|_{\text{BL}}^*$ -norm topology to M .*

Proof. We have to show that for any $\|\cdot\|_{\text{BL}}^*$ -norm closed set C , $C \cap M$ is closed in the restriction of the $\sigma(\mathcal{M}(S), \text{BL}(S))$ -weak topology to M . Since the latter is first countable, $C \cap M$ is relatively $\sigma(\mathcal{M}(S), \text{BL}(S))$ -weak closed if and only if for every $\sigma(\mathcal{M}(S), \text{BL}(S))$ -weakly converging sequence $\mu_n \rightarrow \mu$ in $\mathcal{M}(S)$ with $\mu_n \in C$, one has $\mu \in C$ (cf. [16] Theorem 2.8, p. 72). Let (μ_n) be such a sequence. Because $\sup_{\mu \in M} \|\mu\|_{\text{TV}} < \infty$ by assumption, Theorem 3.2 implies that there exists $\mu' \in \mathcal{M}(S)$ such that $\|\mu_n - \mu'\|_{\text{BL}}^* \rightarrow 0$. Since C is relatively $\|\cdot\|_{\text{BL}}^*$ -norm closed in M , $\mu' \in C$. Moreover, $\langle \mu, f \rangle = \langle \mu', f \rangle$ for every $f \in \text{BL}(S)$, so $\mu = \mu' \in C$. \square

The following technical result provides a tractable condition that ensures first countability of the relative weak topology on the set M , as we shall show after having proven the result. We need to introduce some notation. For $\lambda > 0$ and $C \subset S$ closed and nonempty, define

$$h_{\lambda, C}(x) := \left[1 - \frac{1}{\lambda}d(x, C)\right]^+.$$

Then $h_{\lambda, C} \in \text{BL}(S)$, $|h_{\lambda, C}|_L = \frac{1}{\lambda}$, $0 \leq h_{\lambda, C} \leq 1$ and $h_{\lambda, C} \downarrow \mathbb{1}_C$ pointwise as $\lambda \downarrow 0$. Moreover $h_{\lambda, C} = 0$ on $S \setminus C^\lambda$. We can now state the result.

Lemma 4.1. *Let $M \subset \mathcal{M}(S)$ be such that $m := \sup_{\mu \in M} \|\mu\|_{\text{TV}} < \infty$. If for every $\mu \in M$ and every $\varepsilon > 0$ there exist $K_1, \dots, K_n \subset S$ compact such that for $K = \bigcup_{i=1}^n K_i$:*

$$(i) \quad |\mu|(S \setminus K) < \varepsilon,$$

(ii) *There exists $0 < \lambda_0 \leq \varepsilon$ such that for all $0 < \lambda \leq \lambda_0$ there exists $\delta_1, \dots, \delta_n > 0$ such that the following statement holds:*

$$\text{If } \nu \in M \text{ satisfies } |\langle \mu - \nu, h_{\lambda, K_i} \rangle| < \delta_i \text{ for all } i = 1, \dots, n,$$

$$\text{then } |\nu|(S \setminus K^\lambda) < \varepsilon.$$

Then the relative $\sigma(\mathcal{M}(S), \text{BL}(S))$ -weak topology on M is first countable.

Proof. We first define a countable family \mathcal{F} of functions in $\bar{B} := \{g \in \text{BL}(S) : \|g\|_\infty \leq 1\}$ that is dense in \bar{B} for the compact-open topology, i.e. the topology of uniform convergence on compact subsets of S . Let D be a countable dense subset of S . The family of finite subsets of D is countable. Let $I_{\mathbb{Q}} := \mathbb{Q} \cap [0, 1]$. For a finite subset $F \subset D$, $\lambda \in I_{\mathbb{Q}} \setminus \{0\}$ and function $a : F \rightarrow I_{\mathbb{Q}}$ define

$$f_{F, a}^\lambda(x) := \bigvee_{y \in F} [a(y)(1 - \frac{1}{\lambda}d(x, y))^+].$$

Here \vee denotes the maximum, as before. Then $f_{F,a}^\lambda \in \text{BL}(S)$, $|f_{F,a}^\lambda|_L \leq \max_{y \in F} \frac{a(y)}{\lambda} \leq \frac{1}{\lambda}$. Moreover, $f_{F,a}^\lambda$ vanishes outside $F^\lambda = \bigcup_{y \in F} B(y, \lambda)$. For a finite subset $F \subset D$ the family \mathcal{F}_F of all such functions $f_{F,a}^\lambda$ with a and λ as indicated is countable. So the union \mathcal{F}^+ of all sets \mathcal{F}_F over all finite $F \subset D$ is countable too. It is quickly verified that on any compact subset K of S any positive $h \in \bar{B}$ can be uniformly approximated by $f \in \mathcal{F}^+$. Consequently, $\mathcal{F} = \mathcal{F}^+ - \mathcal{F}^+ \subset \text{BL}(S)$ is countable and any $h \in \bar{B}$ can be approximated uniformly on compact sets by means of $f \in \mathcal{F}$.

Now let $\mu \in M$ and consider the open neighbourhood

$$U_\mu(h, r) := \{\nu \in M : |\langle \mu - \nu, h \rangle| < r\},$$

with $r > 0$ and $h \in \text{BL}(S)$. Without loss of generality we can assume that $\|h\|_{\text{BL}} = 1$. We shall prove that there exist $f_0, \dots, f_n \in \mathcal{F}$ and $q_0, \dots, q_n > 0$ in \mathbb{Q} such that

$$(4.4) \quad \bigcap_{i=0}^n \{\nu \in M : |\langle \mu - \nu, h_i \rangle| < q_i\} \subset U_\mu(h, r).$$

Then the relative weak topology on M is first countable.

Let $\varepsilon \in \mathbb{Q}$ such that $0 < \varepsilon \leq \frac{1}{6}r$ and let $K_i, K \subset S$ be compact and $0 < \lambda_0 \leq \varepsilon$ as in the conditions of the lemma. There exists $f_0 \in \mathcal{F}$ such that $\sup_{x \in K} |h(x) - f_0(x)| \leq \frac{1}{4m}\varepsilon$. Then for any $0 < \lambda \leq \lambda_0$, $x \in K^\lambda$ and $x_0 \in K$,

$$\begin{aligned} |h(x) - f_0(x)| &\leq |h(x) - h(x_0)| + |h(x_0) - f_0(x_0)| + |f_0(x_0) - f_0(x)| \\ &\leq (1 + |f_0|_L)d(x, x_0) + \frac{1}{4m}\varepsilon. \end{aligned}$$

Hence

$$\sup_{x \in K^\lambda} |h(x) - f_0(x)| \leq (1 + |f_0|_L)\lambda + \frac{1}{4m}\varepsilon.$$

Let $0 < \lambda'_0 \leq \lambda_0$ be such that $(1 + |f_0|_L)\lambda'_0 \leq \frac{1}{4m}\varepsilon$. Now one has, using property (i),

$$\begin{aligned} |\langle \mu - \nu, h \rangle| &\leq |\langle \mu - \nu, h - f_0 \rangle| + |\langle \mu - \nu, f_0 \rangle| \\ &\leq \int_{K^\lambda} |h - f_0| d|\mu - \nu| + 2|\mu|(S \setminus K^\lambda) + 2|\nu|(S \setminus K^\lambda) + |\langle \mu - \nu, f_0 \rangle| \\ (4.5) \quad &\leq \frac{1}{2m}\varepsilon \cdot 2m + 2\varepsilon + 2|\nu|(S \setminus K^\lambda) + |\langle \mu - \nu, f_0 \rangle| \end{aligned}$$

for all $0 < \lambda \leq \lambda'_0$. Fix $\lambda \in \mathbb{Q}$ with $0 < \lambda \leq \lambda'_0$ and let $\delta_1, \dots, \delta_n$ be as in property (ii).

The Hausdorff semidistance on closed and bounded subsets of S is given by

$$\delta(C, C') := \sup_{x \in C} d(x, C').$$

The Hausdorff distance is defined by

$$d_H(C, C') := \max(\delta(C, C'), \delta(C', C)).$$

The collection of finite subsets of D form a separable dense subset of the set of compact subsets of S , $\mathcal{K}(S)$, for d_H . If $F \subset D$ is finite and $K' \in \mathcal{K}(S)$, then by the Birkhoff Inequalities

$$\begin{aligned} |h_{\lambda, K'} - h_{\lambda, F}| &= \left| \left[1 - \frac{1}{\lambda} d(x, K') \right]^+ - \left[1 - \frac{1}{\lambda} d(x, F) \right]^+ \right| \\ &\leq \left| \left[1 - \frac{1}{\lambda} d(x, K') \right] - \left[1 - \frac{1}{\lambda} d(x, F) \right] \right| \\ &= \frac{1}{\lambda} |d(x, K') - d(x, F)| \leq \frac{1}{\lambda} \cdot d_H(K', F). \end{aligned}$$

Let $F_i \subset D$ be finite such that $d_H(K_i, F_i) \leq \frac{1}{4m} \lambda \delta_i$. Then $h_{\lambda, F_i} = f_{F_i, \mathbb{1}}^\lambda \in \mathcal{F}$. Put $f_i := h_{\lambda, F_i}$. Let $q_i \in \mathbb{Q}$ be such that $0 < q_i < \frac{1}{2} \delta_i$. If $\nu \in M$ is such that $|\langle \mu - \nu, f_i \rangle| < q_i$ for $i = 1, \dots, n$, then

$$|\langle \mu - \nu, h_{\lambda, K_i} \rangle| \leq \|h_{\lambda, K_i} - h_{\lambda, F_i}\|_\infty \cdot \|\mu - \nu\|_{\text{TV}} + |\langle \mu - \nu, f_i \rangle| < \frac{1}{2} \delta_i + \frac{1}{2} \delta_i = \delta_i$$

According to condition (ii) one has $|\nu|(S \setminus K^\lambda) < \varepsilon$. Put $q_0 = \varepsilon$. Inequality (4.5) then yields (4.4), as desired. \square

Because conditions (i) and (ii) in Lemma 4.1 are immediately satisfied when M is uniformly tight, we obtain

Corollary 4.1. *Let (S, d) be a complete separable metric space and let $M \subset \mathcal{M}(S)$ such that $\sup_{\mu \in M} \|\mu\|_{\text{TV}} < \infty$ and M is uniformly tight. Then the $\sigma(\mathcal{M}(S), \text{BL}(S))$ -weak topology coincides with the $\|\cdot\|_{\text{BL}}^*$ -norm topology on M .*

Remark 4.1. *Gwiazda et al. [9] state at p. 2708 that the topology of narrow convergence in $\mathcal{M}(S)$, i.e. that of convergence of sequences of signed measures paired with $f \in C_b(S)$, is metrizable on tight subsets that are uniformly bounded in total variation norm. In fact it can be metrized by the norm $\|\cdot\|_{\text{BL}}^*$.*

A second case, more involved, in which the conditions of Lemma 4.1 are satisfied, is:

Proposition 4.2. *Let (S, d) be a complete separable metric space and let*

$$M := \{\mu \in \mathcal{M}(S) : \|\mu\|_{\text{TV}} = \rho\}, \quad (\rho > 0).$$

Then condition (i) and (ii) of Lemma 4.1 hold. In particular, the relative $\sigma(\mathcal{M}(S), \text{BL}(S))$ -weak topology and relative $\|\cdot\|_{\text{BL}}^$ -norm topology on M coincide.*

Proof. Take $\varepsilon > 0$, $\mu \in M$ and let μ^+ and μ^- be the positive and negative part of μ , i.e. $\mu = \mu^+ - \mu^-$. Since μ^\pm are disjoint and tight, by Ulam's Lemma, there exist compact sets $K_\pm \subset S$ such that $K_+ \cap K_- = \emptyset$, $\mu^\pm(K_\mp) = 0$ and

$$(4.6) \quad \mu^+(S) - \mu^+(K_+) < \varepsilon/8 \quad \text{and} \quad \mu^-(S) - \mu^-(K_-) < \varepsilon/8.$$

In particular,

$$|\mu|(S \setminus (K_+ \cup K_-)) \leq \mu^+(S \setminus K_+) + \mu^-(S \setminus K_-) < \frac{1}{8}\varepsilon + \frac{1}{8}\varepsilon < \varepsilon,$$

so condition (i) of Lemma 4.1 is satisfied for $K = K_+ \cup K_-$.

Because K_+ and K_- are compact, there exists $\lambda_0 > 0$ such that $K_+^{\lambda_0} \cap K_-^{\lambda_0} = \emptyset$. Then $K_+^\lambda \cap K_-^\lambda = \emptyset$ for all $0 < \lambda \leq \lambda_0$. Without loss of generality we can assume that $\lambda_0 \leq \varepsilon$. Fix $0 < \lambda \leq \lambda_0$.

Let us assume for the moment that $\delta_\pm > 0$ have been selected. At the end we will then see how to choose these, such that condition (ii) will be satisfied. If $\nu \in M$ satisfies

$$(4.7) \quad |\langle \mu - \nu, h_{\lambda, K_+} \rangle| < \delta_+ \quad \text{and} \quad |\langle \mu - \nu, h_{\lambda, K_-} \rangle| < \delta_-,$$

then

$$\langle \mu - \nu^+, h_{\lambda, K_+} \rangle \leq \langle \mu - \nu^+ + \nu^-, h_{\lambda, K_+} \rangle \leq |\langle \mu - \nu, h_{\lambda, K_+} \rangle| < \delta_+.$$

Consequently, since $\mathbb{1}_{K_+} \leq h_{\lambda, K_+} \leq \mathbb{1}_{K_+^\lambda}$,

$$\mu^+(K_+) - \mu^-(K_+^\lambda) - \nu^+(K_+^\lambda) \leq \langle \mu - \nu^+, h_{\lambda, K_+} \rangle < \delta_+.$$

We obtain

$$\begin{aligned} \nu^+(K_+^\lambda) &> \mu^+(K_+) - \mu^-(K_+^\lambda) - \delta_+ \geq \mu^+(K_+) - \mu^-(S \setminus K_-) - \delta_+ \\ &> \mu^+(K_+) - \frac{1}{8}\varepsilon - \delta_+. \end{aligned}$$

In a similar way,

$$\langle -\mu - \nu^-, h_{\lambda, K_-} \rangle \leq \langle \nu - \mu, h_{\lambda, K_-} \rangle < \delta_-,$$

whence

$$\nu^-(K_-^\lambda) > \mu^-(K_-) - \frac{1}{8}\varepsilon - \delta_-.$$

Therefore, using (4.6),

$$\begin{aligned} \nu^+(K_+^\lambda) + \nu^-(K_-^\lambda) &> \mu^+(K_+) + \mu^-(K_-) - \frac{1}{4}\varepsilon - (\delta_+ + \delta_-) \\ &> \mu^+(S) + \mu^-(S) - \frac{1}{2}\varepsilon - (\delta_+ + \delta_-) = \rho - (\delta_+ + \delta_- + \frac{1}{2}\varepsilon). \end{aligned}$$

Note that in this last step the assumption that M is a total variation sphere is used in an essential manner. The last inequality implies that

$$|\nu|(S \setminus K^\lambda) = |\nu|(S) - |\nu|(K_+^\lambda) - |\nu|(K_-^\lambda) \leq \rho - \nu^+(K_+^\lambda) - \nu^-(K_-^\lambda) < \delta_+ + \delta_- + \frac{1}{2}\varepsilon.$$

Thus, if we take $K_1 = K_+$, $K_2 = K_-$, $\delta_+ = \delta_- = \delta_i = \frac{1}{4}\varepsilon$, we see that condition (ii) in Lemma 4.1 is satisfied. Theorem 4.3 then yields the final statement. \square

Remark 4.2. *The result stated in Proposition 4.2 can be found in [24], Corollary 5.39. There, a proof of this result is provided using completely different techniques. Concerning coincidence of these topologies on total variation spheres, see some further notes in [24], indicating e.g. [8].*

In view of Corollary 4.1 and Proposition 4.2 one might be tempted to conjecture that the weak and norm topologies would coincide on sets of measures with uniformly bounded total variation. This does not hold however, as the following counterexample illustrates.

Counterexample. Let (S, d) be the natural numbers \mathbb{N} equipped with the restriction of the Euclidean metric on \mathbb{R} . Now, $\text{BL}(\mathbb{N})$ is linearly isomorphic to ℓ^∞ : the map $f \mapsto (f(n))_{n \in \mathbb{N}}$ is bijective and continuous. Hence it is a linear isomorphism by Banach's Isomorphism

Theorem. Observe that $|f|_L \leq 2\|f\|_\infty$. Since (\mathbb{N}, d) is uniformly discrete, the norms $\|\cdot\|_{\text{BL}}^*$ and $\|\cdot\|_{\text{TV}}$ on $\mathcal{M}(\mathbb{N})$ are equivalent (cf. [14], proof of Theorem 3.11). So $\mathcal{M}(\mathbb{N})_{\text{BL}}$ is linearly isomorphic to ℓ^1 under the map $\mu \mapsto (\mu(\{n\}))_{n \in \mathbb{N}}$. One has $\|\mu\|_{\text{TV}} = \|(\mu)\|_{\ell^1}$. Moreover, the duality between $\mathcal{M}(\mathbb{N})$ and $\text{BL}(\mathbb{N})$ is precisely the duality between ℓ^1 and ℓ^∞ under the given isomorphisms. Consider now $M := \{(\mu) \in \ell^1 : \|(\mu)\|_{\ell^1} \leq 1\}$. It represents a set of measures that is uniformly bounded in total variation norm. Let $S := \{(\mu) \in \ell^1 : \|(\mu)\|_{\ell^1} = 1\}$. Then S is a $\|\cdot\|_{\text{TV}}$ -closed subset of M . The weak closure of S equals M however (cf. [6], Section V.1, Ex. 10). Therefore, the $\|\cdot\|_{\text{BL}}^*$ (i.e. $\|\cdot\|_{\text{TV}}$) and weak topologies cannot coincide on M .

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MATHEMATICAL INSTITUTE, LEIDEN UNIVERSITY, P.O. BOX 9512, 2300 RA LEIDEN, THE NETHERLANDS, (SH, MZ)

E-mail address: {shille,m.a.ziemplanska}@math.leidenuniv.nl

FACULTY OF APPLIED PHYSICS AND MATHEMATICS, GDAŃSK UNIVERSITY OF TECHNOLOGY, GABRIELA NARUTOWICZA 11/12, 80-233 GDAŃSK (TS)

E-mail address: szarek@intertele.pl

TNO, CYBER SECURITY AND ROBUSTNESS, P.O. BOX 96800, 2509 JE, THE HAGUE, THE NETHERLANDS, (DW)

E-mail address: daniel.worm@tno.nl